Entering the tower with Iwasawa theory

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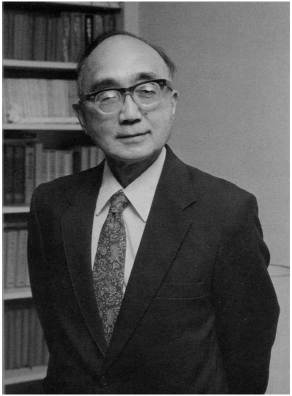
General idea about Iwasawa theory

Iwasawa theory was born as the study of the growth of the ideal class group of $\mathbb{Q}(\zeta_{p^n})$ over towers of numbers fields.

The three main characteristics of (general) Iwasawa theory:

- Studying the growth of objects of arithmetic nature...
- ...over infinite towers of fields....
- ... which are built using p-adic extensions.



Kenkichi Iwasawa. 1917-1998

Fermat's Last Theorem (Wiles, 1995)

The equation $x^n + y^n = z^n$ has no non-trivial solutions for every integer $n \geq 3$.

Around 1840, Kummer developed his theory of cyclotomic fields trying to prove nice properties of the complex factorization of

$$x^p + y^p = \prod_{i=0}^{p-1} (x + \zeta_p^i y)$$

in the ring $\mathbb{Z}[\zeta_p]$, where ζ_p is the p-th root of unity.

Ernst Kummer. 1810-1893

Problem: $\mathbb{Z}[\zeta_p]$ is not a principal ideal domain in general!

- Kummer defines the **ideal class group** $Cl(K)$ of a number field K, which measures the failure of the ring of integers $\mathcal{O}_K$ of K to be a PID.

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- Reminder: A finite group G has a p-Sylow subgroup for every prime p, which consists of all the elements of G whose order is a power of p.

A miraculous connection!

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The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\ell \text{ prime}} \frac{1}{1-\ell^{-s}},$$

and the p-Sylow subgroup of the ideal class group are deeply related!

Kummer's criterion

A prime p is irregular (i.e. the p-Sylow subgroup of $Cl(\mathbb{Q}(\zeta_p))$ *is non-trivial) if and only if p divides the numerator of at least one of* $\zeta(-1), \zeta(-3), \ldots, \zeta(4-p)$.

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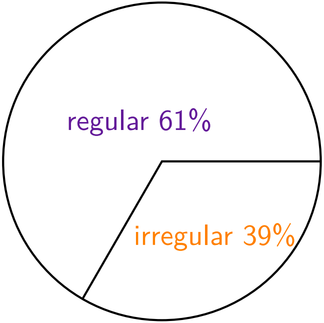

First irregular primes: 37, 59, 67, 101, 103...

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Kubota-Leopoldt p-adic L-function, 1964

Fix $k \in \mathbb{Z}$. There exists a continuous $\mathbb{Z}_p$-valued function $L_p(\omega^k, s)$ of p-adic variable $s \in \mathbb{Z}_p$ satisfying

$$L_p(\omega^k, 1-n) = (1-p^{n-1})\zeta(1-n)$$

for all $n \equiv k \bmod p-1$, where ω is the p-adic character

$$\omega \colon (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p.$$

p-adic world

- We define the p-**adic integers** as an inverse limit

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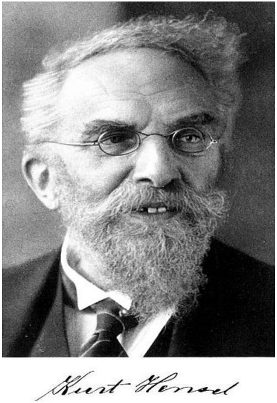

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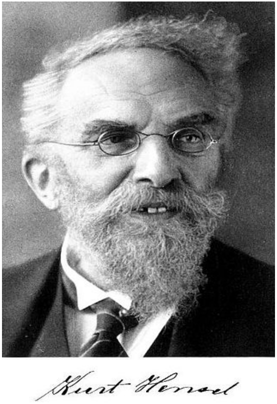

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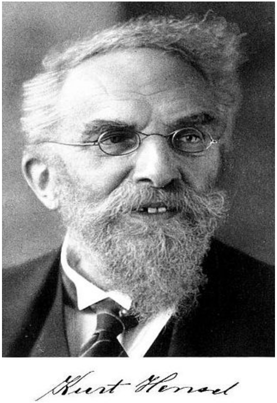
- $(a \bmod p, a \bmod p^2, a \bmod p^3, \dots) \in \mathbb{Z}_p$
- Taking the fraction field of $\mathbb{Z}_p$, we get the p-**adic field** $\mathbb{Q}_p$.



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- Completing $\mathbb{Q}$ with respect to the p-adic absolute value, we obtain a complete local field $\mathbb{Q}_p$.
- Now, the p-adic integers are
$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Why should we care about the p-adics?

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- **(Hasse principle or local-to-global principle)** An equation has a solution over $\mathbb{Q}$ if and only if it has a solution over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes p. Not true in general!

$\mathbb{Z}_p$-extensions

- Take all extensions K_n/K such that

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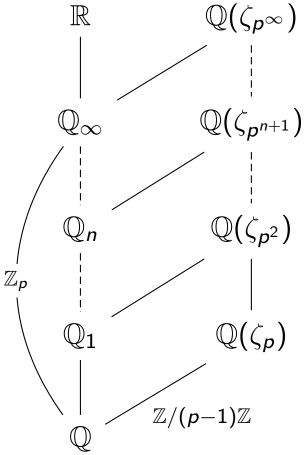
- $\mathbb{Q}_\infty$ is the *only* $\mathbb{Z}_p$-extension of $\mathbb{Q}$, called the **cyclotomic** $\mathbb{Z}_p$-extension.



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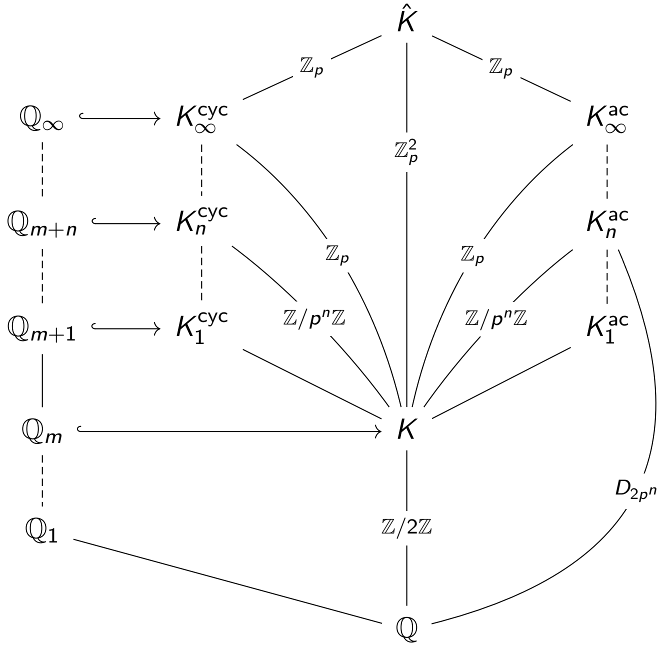
Leopoldt's conjecture

With the same notation as above, $d = r_2 + 1$.

Proven for abelian extensions $K/\mathbb{Q}$ by Brumer in 1976.

Example: K imaginary quadratic field.



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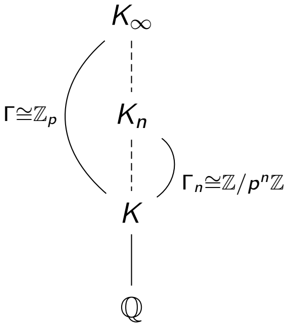
- Stronger: it is a module over the Iwasawa algebra Λ.

Iwasawa algebra

Notation: $\Gamma = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$, $\Gamma_n = \mathrm{Gal}(K_n/K) \cong \mathbb{Z}/p^n\mathbb{Z}$.



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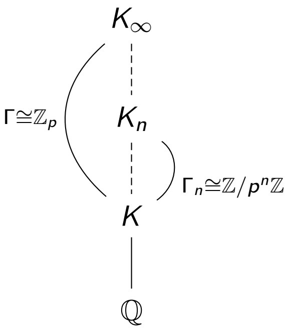

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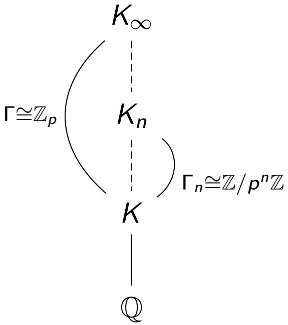
where γ is a topological generator of Γ.



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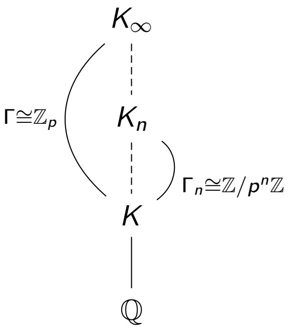

A monic polynomial $f(T) \in \mathbb{Z}_p[T]$ is called *distinguished* if all its coefficients (except the leading) are divisible by p.

Structure theorem for f.g. Λ-modules. Iwasawa, Serre.

Let M be a finitely generated Λ-module. Then

$$M \sim \Lambda^{rank} \oplus \bigoplus_{i=1}^{r} \Lambda/(f_i^{k_i}) \oplus \bigoplus_{j=1}^{s} \Lambda/(p^{m_j})$$

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Iwasawa's Main Conjecture. Theorem by Mazur-Wiles.

$$\mathrm{char}(A_\infty) = (L_p).$$

arithmetic objects $\leftrightsquigarrow$ L-functions

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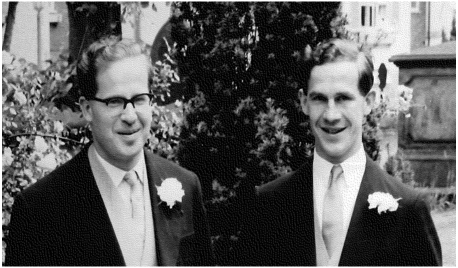
- **Open question**: possibilities for $\text{rank}(E(\mathbb{Q}))$?

A Millennium Prize Problem...



Birch and Swinnerton-Dyer conjecture

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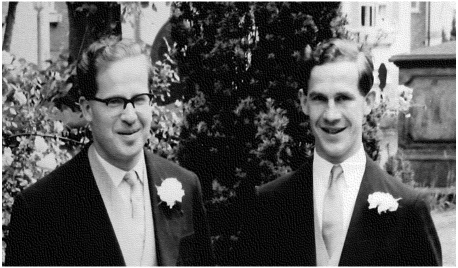

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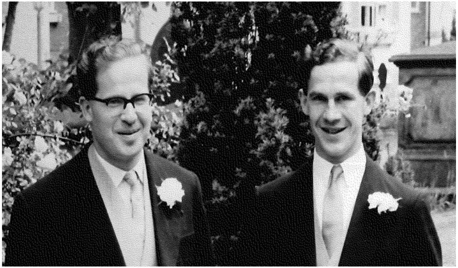

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Let $E/\mathbb{Q}$ be an elliptic curve and $L(E,s)$ be its L-function.

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- *Residue of $L(E,s)$ at $s=1$:*

$$\lim_{s\to 1} \frac{L(E,s)}{(s-1)^{R_E}} = \frac{|Ш_E| \cdot \Omega_E \cdot \operatorname{Reg}(E/\mathbb{Q}) \cdot \prod_p c_p}{|E_{\text{tors}}(\mathbb{Q})|^2}.$$

Selmer groups

Setting: E/K elliptic curve with good ordinary reduction at all primes above p, where $K_\infty = \bigcup_n K_n$ is the *cyclotomic* $\mathbb{Z}_p$-extension of K. Assume that the Tate-Shafarevich group $\text{Ш}_E(K)$ is finite.

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Idea: study the growth of $\mathrm{Sel}_E(K_n)_p$ over K_∞.

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The natural maps

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Corollary: Assume that $E(K_n)$ is finite for all n. There are non-negative $\lambda, \mu, \nu \in \mathbb{Z}$ such that

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Theorem. Kato, Rohrlich

Assume $K = \mathbb{Q}$. Then rank$(E(K_n))$ is bounded and independent of n.

- **Pontryagin dual** of $\mathrm{Sel}_E(K_\infty)_p$:

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Main Conjecture for Elliptic Curves

$$\mathrm{char}(X_E(K_\infty)) = (L_p(E,s)).$$

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Thank you!!